Section 6.3
Polar Coordinates

Getting a New Perspective

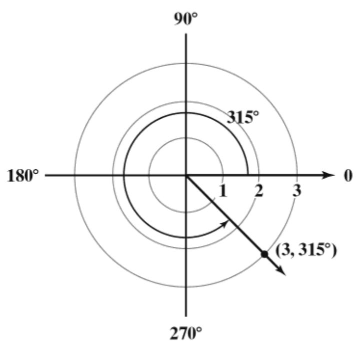
We have worked extensively in the Cartesian coordinate system, plotting points, graphing equations, and using the properties of the Cartesian plane to investigate functions and solve problems. In this section, we introduce a different coordinate system that will give you a new perspective of the plane. In this system, points do not have unique ordered pairs associated with them and some complicated-looking graphs have very simple equations.

Objective #1: Plot points in the polar coordinate system.

✔ *Solved Problem #1*

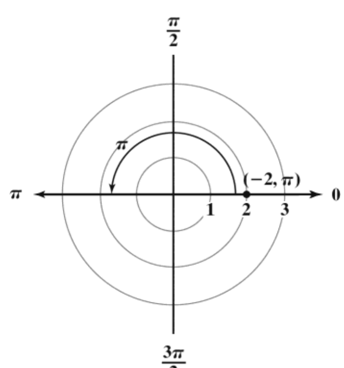
1a. Plot the point with polar coordinates (3, 315°).

Draw an angle measuring 315° in standard position. Then plot a point 3 units from the origin (pole) along the terminal side of the angle.



1b. Plot the point with polar coordinates (–2, π).

Draw an angle measuring π radians in standard position. Then plot a point 2 units from the origin (pole) along a line in the opposite direction from the terminal side of the angle. Since the terminal side of the angle points to the left, this means that we move 2 units to the right.



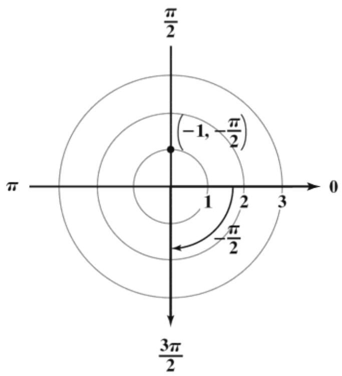
✎ *Pencil Problem #1* ✎

1a. Plot the point with polar coordinates (2, 45°).

1b. Plot the point with polar coordinates (–1, π).

1c. Plot the point with polar coordinates $\left(-1,\ -\frac{\pi}{2}\right)$.

Draw an angle measuring $-\frac{\pi}{2}$ radians in standard position. Then plot a point 1 unit from the origin (pole) along a line in the opposite direction from the terminal side of the angle. Since the terminal side of the angle points down, this means that we move 2 units up.



1c. Plot the point with polar coordinates $\left(-2,\ -\frac{\pi}{2}\right)$.

Objective #2: Find multiple sets of polar coordinates for a given point.

✔ ***Solved Problem #2***

2a. Find a representation of $\left(5,\ \frac{\pi}{4}\right)$ in which r is positive and $2\pi < \theta < 4\pi$.

Add 2π to the angle and do not change r.

$$\left(5,\ \frac{\pi}{4}\right) = \left(5,\ \frac{\pi}{4} + 2\pi\right) = \left(5,\ \frac{\pi}{4} + \frac{8\pi}{4}\right) = \left(5,\ \frac{9\pi}{4}\right)$$

✎ ***Pencil Problem #2*** ✎

2a. Find a representation of $\left(10,\ \frac{3\pi}{4}\right)$ in which r is positive and $2\pi < \theta < 4\pi$.

2b. Find a representation of $\left(5, \frac{\pi}{4}\right)$ in which r is negative and $0 < \theta < 2\pi$.

Add π to the angle and replace r with $-r$.

$$\left(5, \frac{\pi}{4}\right) = \left(-5, \frac{\pi}{4} + \pi\right) = \left(-5, \frac{\pi}{4} + \frac{4\pi}{4}\right) = \left(-5, \frac{5\pi}{4}\right)$$

2b. Find a representation of $\left(10, \frac{3\pi}{4}\right)$ in which r is negative and $0 < \theta < 2\pi$.

2c. Find a representation of $\left(5, \frac{\pi}{4}\right)$ in which r is positive and $-2\pi < \theta < 0$.

Subtract 2π from the angle and do not change r.

$$\left(5, \frac{\pi}{4}\right) = \left(5, \frac{\pi}{4} - 2\pi\right) = \left(5, \frac{\pi}{4} - \frac{8\pi}{4}\right) = \left(5, -\frac{7\pi}{4}\right)$$

2c. Find a representation of $\left(10, \frac{3\pi}{4}\right)$ in which r is positive and $-2\pi < \theta < 0$.

Objective #3: Convert a point from polar to rectangular coordinates.

✔ *Solved Problem #3*

3a. Find the rectangular coordinates of the point with polar coordinates $(3, \pi)$.

$$x = r\cos\theta = 3\cos\pi = 3(-1) = -3$$
$$y = r\sin\theta = 3\sin\pi = 3(0) = 0$$

The rectangular coordinates are $(-3, 0)$.

Pencil Problem #3

3a. Find the rectangular coordinates of the point with polar coordinates $\left(2, \frac{\pi}{3}\right)$.

3b. Find the rectangular coordinates of the point with polar coordinates $\left(-10, \frac{\pi}{6}\right)$.

$x = r\cos\theta = -10\cos\frac{\pi}{6} = -10\left(\frac{\sqrt{3}}{2}\right) = -5\sqrt{3}$

$y = r\sin\theta = -10\sin\frac{\pi}{6} = -10\left(\frac{1}{2}\right) = -5$

The rectangular coordinates are $(-5\sqrt{3}, -5)$.

3b. Find the rectangular coordinates of the point with polar coordinates $\left(-4, \frac{\pi}{2}\right)$.

Objective #4: Convert a point from rectangular to polar coordinates.

✔ *Solved Problem #4*

4a. Find polar coordinates of the point with rectangular coordinates $(1, -\sqrt{3})$.

The point $(1, -\sqrt{3})$ is in quadrant IV. Find r.

$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$

Find θ. Since $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ and θ lies in quadrant IV,

$\theta = 2\pi - \frac{\pi}{3} = \frac{6\pi}{3} - \frac{\pi}{3} = \frac{5\pi}{3}$.

The polar coordinates are $\left(2, \frac{5\pi}{3}\right)$.

✎ *Pencil Problem #4* ✎

4a. Find polar coordinates of the point with rectangular coordinates (–2, 2).

4b. Find polar coordinates of the point with rectangular coordinates (0, –4).

The point (0, –4) lies on the negative *y*-axis 4 units below the origin. Thus, $r = 4$ and $\theta = \frac{3\pi}{2}$.

The polar coordinates are $\left(4, \frac{3\pi}{2}\right)$.

4b. Find polar coordinates of the point with rectangular coordinates (5, 0).

Objective #5: Convert an equation from rectangular to polar coordinates.

✔ *Solved Problem #5*

5a. Covert $3x - y = 6$ to a polar equation.

Replace *x* with $r\cos\theta$ and *y* with $r\sin\theta$ and solve for *r*.

$$3x - y = 6$$
$$3r\cos\theta - r\sin\theta = 6$$
$$r(3\cos\theta - \sin\theta) = 6$$
$$r = \frac{6}{3\cos\theta - \sin\theta}$$

Pencil Problem #5

5a. Covert $3x + y = 7$ to a polar equation.

5b. Covert $x^2 + (y+1)^2 = 1$ to a polar equation.

Replace *x* with $r\cos\theta$ and *y* with $r\sin\theta$ and solve for *r*.

$$x^2 + (y+1)^2 = 1$$
$$(r\cos\theta)^2 + (r\sin\theta+1)^2 = 1$$
$$r^2\cos^2\theta + r^2\sin^2\theta + 2r\sin\theta + 1 = 1$$
$$r^2(\cos^2\theta + \sin^2\theta) + 2r\sin\theta = 0$$
$$r^2 + 2r\sin\theta = 0$$
$$r(r + 2\sin\theta) = 0$$
$$r = 0 \quad \text{or} \quad r + 2\sin\theta = 0$$
$$r = -2\sin\theta$$

The graph of $r = 0$ is the pole. The graph of $r = -2\sin\theta$ also includes the pole, so it is not necessary to include the equation $r = 0$.

The polar equation is $r = -2\sin\theta$.

5b. Covert $(x-2)^2 + y^2 = 4$ to a polar equation.

Objective #6: Convert an equation from polar to rectangular coordinates.

Solved Problem #6

6a. Convert $r = 4$ to a rectangular equation.

Square each side in anticipation of using $r^2 = x^2 + y^2$.

$$r = 4$$
$$r^2 = 4^2$$
$$x^2 + y^2 = 16$$

The rectangular equation is $x^2 + y^2 = 16$, which we recognize as the equation of a circle centered at the origin with radius 4.

Pencil Problem #6

6a. Convert $r = 8$ to a rectangular equation.

6b. Convert $\theta = \frac{3\pi}{4}$ to a rectangular equation.

Take the tangent of each side and use $\tan\theta = \frac{y}{x}$.

Then solve for y.

$$\tan\theta = \tan\frac{3\pi}{4}$$
$$\frac{y}{x} = -1$$
$$y = -x$$

The rectangular equation is $y = -x$, which we recognize as the equation of a line passing through the origin with slope –1.

6b. Convert $\theta = \frac{\pi}{2}$ to a rectangular equation.

6c. Convert $r = -2\sec\theta$ to a rectangular equation.

Use a reciprocal identity to rewrite the secant in terms of cosine. Multiply each side by $\cos\theta$ and replace $r\cos\theta$ with x.

$$r = -2\sec\theta$$
$$r = \frac{-2}{\cos\theta}$$
$$r\cos\theta = -2$$
$$x = -2$$

The rectangular equation is $x = -2$, which we recognize as the equation of a vertical line with x-intercept –2.

6c. Convert $r = 4\csc\theta$ to a rectangular equation.

6d. Convert $r = 10\sin\theta$ to a rectangular equation.

Multiply both sides by r and then replace r^2 with $x^2 + y^2$ and $r\sin\theta$ with y.

$$r = 10\sin\theta$$
$$r^2 = 10r\sin\theta$$
$$x^2 + y^2 = 10y$$
$$x^2 + y^2 - 10y = 0$$
$$x^2 + (y^2 - 10y + 25) = 25$$
$$x^2 + (y - 5)^2 = 25$$

The rectangular equation is $x^2 + (y - 5)^2 = 25$, which we recognize as the equation of a circle centered at (0, 5) with radius 5.

6d. Convert $r = 12\cos\theta$ to a rectangular equation.