Section 10.5
The Binomial Theorem

Who Knew That First?

Telephones, Internet, and other modern forms of communication mean that information now can spread across the globe in the blink of an eye.

In this section, we study a special array of numbers known as Pascal's triangle, named after mathematician Blaise Pascal. However, this triangular array of numbers actually appeared centuries earlier in a Chinese document.

The same mathematics is often discovered by independent researchers separated by time, place, and culture. But with modern communication, important discoveries are now shared much more efficiently.

Objective #1: Evaluate a binomial coefficient.

✔ *Solved Problem #1*

1a. Evaluate: $\binom{6}{3}$.

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = \frac{6\cdot 5\cdot 4\cdot \cancel{3!}}{3\cdot 2\cdot 1\cdot \cancel{3!}} = 20$$

1b. Evaluate: $\binom{6}{0}$.

$$\binom{6}{0} = \frac{6!}{0!(6-0)!} = \frac{6!}{6!} = 1$$

✎ *Pencil Problem #1* ✎

1a. Evaluate: $\binom{8}{3}$.

1b. Evaluate: $\binom{12}{1}$.

1c. Evaluate: $\binom{8}{2}$.

$$\binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8!}{2!6!} = \frac{8 \cdot 7}{2} = 28$$

1c. Evaluate: $\binom{100}{2}$.

1d. Evaluate: $\binom{3}{3}$.

$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = \frac{3!}{3!0!} = \frac{3!}{3!} = 1$$

1d. Evaluate: $\binom{6}{6}$.

Objective #2: Expand a binomial raised to a power.

✔ *Solved Problem #2a*

2a. Expand: $(x+1)^4$

$$(x+1)^4 = \binom{4}{0}x^4 + \binom{4}{1}x^3 + \binom{4}{2}x^2 + \binom{4}{3}x + \binom{4}{4} = x^4 + 4x^3 + 6x^2 + 4x + 1$$

Pencil Problem #2a

2a. Expand: $(x+2)^3$

✔ *Solved Problem #2b*

2b. Expand: $(x-2y)^5$

$$
\begin{aligned}
(x-2y)^5 &= \binom{5}{0}x^5(-2y)^0+\binom{5}{1}x^4(-2y)^1+\binom{5}{2}x^3(-2y)^2+\binom{5}{3}x^2(-2y)^3+\binom{5}{4}x(-2y)^4+\binom{5}{5}x^0(-2y)^5\\
&= x^5 \quad -5x^4(2y) \quad +10x^3(4y^2) \quad -10x^2(8y^3) \quad +5x(16y^4) \quad -32y^5\\
&= x^5-10x^4y+40x^3y^2-80x^2y^3+80xy^4-32y^5
\end{aligned}
$$

Pencil Problem #2b

2b. Expand: $\left(x^2+2y\right)^4$

Objective #3: Find a particular term in a binomial expansion.

✔ ***Solved Problem #3***

3. Find the fifth term in the expansion of $(2x+y)^9$.

Since we are looking for the 5^{th} term, $r = 5 - 1 = 4$.
Thus, $r = 4$, $a = 2x$, $b = y$, and $n = 9$.

$$(r+1)\text{st term} = \binom{n}{r} a^{n-r} b^r$$

$$\text{fifth term} = \binom{9}{4} (2x)^5 y^4$$

$$= \frac{9!}{4!5!}(32x^5)y^4$$

$$= 4032x^5y^4$$

Pencil Problem #3

3. Find the sixth term in the expansion of $\left(x^2+y^3\right)^8$.

Answers for Pencil Problems *(Textbook Exercise references in parentheses)*:

1a. 56 *(10.5 #1)*

1b. 12 *(10.5 #3)*

1c. 4950 *(10.5 #7)*

1d. 1 *(10.5 #5)*

2a. $x^3 + 6x^2 + 12x + 8$ *(10.5 #9)*

2b. $x^8 + 8x^6y + 24x^4y^2 + 32x^2y^3 + 16y^4$ *(10.5 #17)*

3. $56x^6y^{15}$ *(10.5 #43)*