Section 10.4
Mathematical Induction

Will They ALL Fall Down?

The mathematical principle of this section can be illustrated using an unending line of dominoes. If the first domino is pushed over, it knocks down the next, which knocks down the next, and so on, in a chain reaction.

To topple all the dominoes in the infinite sequence, two conditions must be satisfied:

1. The first domino must be knocked down.

2. If the domino in position k is knocked down, then the domino in position $k+1$ must be knocked down.

If the second condition is not satisfied, it does not follow that all the dominoes will topple. For example, suppose the dominoes are spaced far enough apart so that a falling domino does not push over the next domino in the line.

Objective #1: Understand the principle of mathematical induction.

✔ *Solved Problem #1*

1a. For the given statement S_n, write the statement S_1.

$$S_n : 2+4+6+\cdots+2n = n(n+1)$$

If $n=1$ then the statement S_1 is obtained by writing the first term, 2, on the left, and substituting 1 for n on the right.

$S_1 : 2 = 1(1+1)$

✎ *Pencil Problem #1* ✎

1a. For the given statement S_n, write the statement S_1.

$$S_n : 1+3+5+(2n-1) = n^2$$

✔ *Solved Problem #1b*

1b. For the given statement S_n, write the two statements S_k, and S_{k+1}.

$$S_n: 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Write S_k by taking the sum of the first *k* terms on the left and replacing *n* with *k* on the right.

$$S_k: 1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$$

Write S_{k+1} by taking the sum of the first $k+1$ terms on the left and replacing *n* with $k+1$ on the right.

$$S_{k+1}: 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+1+1)^2}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

✎ *Pencil Problem #1b* ✎

1b. For the given statement S_n, write the two statements S_k, and S_{k+1}.

$$S_n: 3 + 7 + 11 + \cdots + (4n-1) = n(2n+1)$$

Objective #2: Prove statements using mathematical induction.

✔ *Solved Problem #2*

2. Use mathematical induction to prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for all positive integers *n*.

Step 1. *Show that* S_1 *is true*:

$$1^3 = \frac{1^2(1+1)^2}{4}$$

$$1 = \frac{1(2)^2}{4}$$

$$1 = \frac{4}{4}$$

$$1 = 1, \text{ True}$$

Step 2. *Show that if* S_k *is true, then* S_{k+1} *is true*:

Assume $1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$ is true. Then,

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2\left(k^2 + 4(k+1)\right)}{4}$$

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2\left(k^2 + 4k + 4\right)}{4}$$

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

The final statement is S_{k+1}.

Thus, by mathematical induction, the statement $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ is true for all positive integers *n*.

✎ *Pencil Problem #2* ✐

2. Use mathematical induction to prove that $1+2+2^2+\cdots+2^{n-1}=2^n-1$ for all positive integers n.

